Production of Commodities by Means of Commodities[1]

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This book is remarkable on many counts. It is handsomely produced and extraordinarily terse, accomplishing a great deal in a tiny space. It is essentially and explicitly mathematical, yet gives few adequate proofs and no mathematical references (apart from the eminent mathematicians whose help is acknowledged in the preface). It is clearly tangential at several points to much modern work on general equilibrium models, but no guide is given to the similarities or differences. And, as perhaps befits a work that is partly concerned with capital theory, it has taken a very long time to come to fruition. Sraffa tells us in his preface that the "central propositions had taken shape in the late 1920's". It is an absorbing game to speculate how much more progress economic theory would have made if those propositions had been published then, before many of today's leading theorists were born. For, with all its oddities, the book is obviously the work of an artist, working in the medium of economic theory.

It is my experience that some economists have concluded that the book gives just another Leontief-type model, subjectively original but nothing more; while others have hailed it as a great advance[2]. Part of this divergence of views is no doubt due to differences in the approaches of various schools of thought, but part must also be due to the book's extreme difficulty. Compressed and mathematically incomplete as it is, the main trouble lies not there, but in wrenching oneself out of the more usual Walrasian approach to general equilibrium, and in substituting a Ricardian viewpoint. An added potential source of difficulty stems from Sraffa's elegant but formal method of presentation. Quite detailed propositions are derived from a model that apparently has almost no assumptions about reality at all, a methodological procedure that is uncommon in modern economic theory (though not necessarily the worse for that).

Because of these difficulties, and because previous review articles have dealt ably with other aspects of Sraffa's contribution, it has seemed to me that the most

[1] A critique of Piero *Sraffa:* Production of Commodities by Means of Commodities; Cambridge, at the University Press, 1960, pp. xii+99. Price 12s 6d.

I am grateful to Hugh *Rose* for several illuminating conversations on the nature of Mr. Sraffa's system.

[2] For a typical reaction of the first type, see R. *Quandt*'s review in Journal of Political Economy 69, October 1961, p. 500; and for the second type of judgment, see Joan *Robinson*, Oxford Economic Papers 13, February 1961, pp. 53–58, and R. L. *Meek*, Scottish Journal of Political Economy 8, June 1961, pp. 119–136. An intermediate position is taken by M. W. *Reder*, American Economic Review 51, September 1961, pp. 688 to 695.

useful function that this critique can serve is to *translate* his work into the more widely used Walrasian dialect of mathematical economics, and to give proofs of his main results which are acceptable to the speakers of that dialect. Translated into this more common *argot,* his system may become less opaque, although perhaps–as in good poetry–there are subtleties which defy translation; at least it seems worth trying. Because of space limitations, I shall be almost entirely concerned with Part I of the book. Once this first half has been mastered, then the rest of the book–which contains some brilliant pieces of analysis–is relatively easy territory, and may be left to the reader to explore.

I.

The system discussed in Chapter 1 (which consists of two and a half pages) is considerably different from the usual static inter-industry model, which superficially it resembles. It is obtained from the following assumptions:

Assumption 1. The economy is producing sufficient of each commodity to maintain itself, or is in what Sraffa calls a *self-replacing state.* This means that we are *given* total quantities of each product, and discuss the conditions under which these quantities will persist. No question of unemployment of any factor can arise, because that would imply schedules for factor supplies, which do not exist in the model.

Assumption 2. We are given certain *numbers* which tell us how much of each commodity is required to produce a *given* quantity of any one commodity (this quantity being given by Assumption 1). The collection of such numbers for any given commodity simply constitutes a *recipe,* as in a cookery-book, and no question of variation in either input proportions or *scale* of output is raised. Hence all we have for any product is one point of its production function; in particular, this is *not* the fixed coefficient model of Walras-Cassel-Leontief type which it appears to resemble.

Assumption 3. There is no surplus in the production of any commodity, i. e. all the output of each product is used to produce other products (including itself), and none goes for final consumption. To make this palatable, we must assume that workers are produced like any other commodity, requiring definite inputs of wheat and wine, etc. in order to produce the *given* amount of labor time postulated by Assumption 1.

As in all such models, we have to make some assumptions about the organization of the market. Sraffa assumes, in effect, that the market is cleared once each production period ("after the harvest")–which therefore is implicitly assumed to be the same for each commodity, a very important proviso. After the exchange takes place, all economizing activity ceases until the next "market day". The only activity pursued between whiles is production, and that goes by the simple recipes of Assumption 2, leaving no room for choice, regarding either input substitution *or* scale.

The model may now be formalized, in a notation differing from–and more convenient than–that used by Sraffa. Assumption 1 says that for each product j there is a *fixed* quantity A_j, so that we may take A_j as the *unit* of measurement of the jth good; we put $A_j = 1$ for each j (and shall assume that there are n commodities). Assumption 2 then permits us to write $a_{i1}, a_{i2}, \ldots, a_{in}$ as the *proportion* of the output of the 1st, 2nd, ..., nth good respectively, used in the production of good i. Assumption 3 tells us that $a_{1j} + a_{2j} + \ldots + a_{nj} = 1$, for each j.

The question we ask of this system (which we denote by $S1$) is: What set (if any) of exchange values relating the various products would enable production to persist indefinitely at the levels prescribed by Assumption 1? Each industry brings one unit of its product to market "after the harvest" and by means of exchange tries to secure just those amounts of each of its inputs required to produce "next year" one unit of product. Can a set of consistent exchange-ratios be found? Notice that $S1$ cannot really be asked any *other* question, since there is no guide to tell us what might happen to *anything* if the levels of production of each commodity were different from 1 (cf. the first paragraph of Sraffa's preface, p.v).

We may rephrase the question as follows: Does there exist a vector of positive exchange ratios $p_1, p_2, \ldots, p_n$–one for each commodity–such that the value of the quantities of the products used in producing each of the unit levels is equal to the value of the (unit) output of each commodity? Symbolically, is there a positive vector $\mathbf{p}^*$ which is a solution of the matrix equation:

$$\mathbf{A}\mathbf{p} = \mathbf{p}, \tag{1}$$

where $\mathbf{A}$ is the matrix of the proportions a_{ij}, and $\mathbf{p}$ is the vector of exchange ratios? Since $\sum_i a_{ij} = 1$ for each j, and $a_{ij} \geqq 0$ for each element of $\mathbf{A}$, this model is formally identical with the linear exchange model introduced by *Remak*, and analyzed in great detail by *Gale*[1].

[1] R. *Remak:* Kann die Volkswirtschaftslehre eine exakte Wissenschaft werden?, Jahrbücher für Nationalökonomie und Statistik *76*, 1929, pp. 703–735. David *Gale:* The Theory of Linear Economic Models (McGraw Hill, 1960), Chapter 8, pp. 260–271. The present model could also be treated by well-known techniques applicable to Markov (or stochastic) matrices, since **A** belongs to this type. See e.g. F. R. *Gantmacher:* Applications of the Theory of Matrices (Interscience, 1959), Chapter III, pp. 99–117.

Unless otherwise stated, all summation signs in this article will be over the n indices $1, 2, \ldots, n$.

Briefly stated, the answer is that there will always exist a solution vector $\mathbf{p}^*$ which will have *no* negative elements and at least one positive element (Gale employs the useful term "semipositive"). This solution will be unique except for a scale factor, which in this model is equivalent to requiring that one of the goods be selected as numéraire. In order to go beyond this statement, and assert the existence of a completely positive vector of exchange-ratios, we have to make a more detailed investigation.

Suppose that there were only three industries, and that the output of one of them–number 3–was not used by either of the others. The relevant matrix equation would then be

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}. \qquad (2)$$

By putting the bottom row to the top, and then the right hand column to the left, we can rewrite (2) as

$$\begin{bmatrix} a_{33} & a_{31} & a_{32} \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} p_3 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_3 \\ p_1 \\ p_2 \end{bmatrix}. \qquad (2\,\text{a})$$

Because of Assumption (3), $a_{33} = 1$, and in view of (2a) we must therefore have $p_3 + a_{31}p_1 + a_{32}p_2 = p_3$. Now a_{31} and a_{32} are both positive, by hypothesis, so that $p_1 = p_2 = 0$, in which case p_3 can assume *any* value. Since, presumably, if an industry in this system $S1$ produces a free good, it goes out of existence, the requirement that the system be in a self-replacing state *implies* positive prices, and hence that the phenomenon discussed here cannot happen. This means in turn that *each* commodity must be used directly or indirectly in the production of every commodity. It is then a theorem (e.g., *Gale*, op. cit., p.226), that there exists a positive price vector, unique save for specification of numéraire[1].

II.

It is possible to move from the system $S1$ in a number of different directions. One such direction would be to assume that the proportions $a_{i1}, a_{i2}, \ldots, a_{in}$

[1] *Sraffa* cites only the equality of equations and unknowns in (1) to guarantee the existence of a *positive* price vector. But it is clear that he is fully aware of these problems, both from his footnote (concerning non-viability), on p. 5, and from his assertion on p. 7 that non-basics (to be defined below) cannot appear in $S1$. This is an example of what was meant above by saying that the book is mathematically (though not economically) incomplete. This property has misled one reviewer (*Meek*, op. cit., p. 119) into saying that "the mathematics used is of a very elementary character", which is akin to equating an iceberg to that part of it which shows above water.

remain invariant with changes in the scale of output, for any commodity i. This would lead directly to the Walras-Cassel-Leontief fixed-coefficient model already mentioned. If then alternative ways of producing any given commodity were introduced, we would be in one region of the world of Koopmans' activity analysis; a further step would take us to the von Neumann model of an expanding economy[1].

Sraffa's direction is quite different, and is related (though not by descent) to his interpretation of an important passage in Ricardo's thought[2]. If the economy is split into two sectors, agriculture ("corn") and manufacture ("iron"), then corn necessarily enters into iron production (through its role as wage good) and also into its own production; but iron does not enter into corn production. If the economy is capable of generating a surplus, then Ricardo seems to have maintained that it necessarily follows that the rate of profit is determined solely by the physical conditions of corn production, and that the exchange-value of iron is independent of the conditions of corn production.

If we consider a more complicated model, with more than two sectors, then can we continue to make the same kind of statement? This is a natural extension of Ricardian ideas in one direction, while a further generalization would be the abandonment of wage determination through the Malthusian population mechanism. Finally, to take in another strand of the Ricardian fabric, it would be desirable to generalize the idea of a commodity which is a "standard of value" i.e. a commodity whose cost of production (and therefore price) is invariant through time, and hence which can serve as an unvarying yardstick against which to measure price changes in other commodities. A rise in the price of a good Y relative to this "fixed" commodity X, would then be unambiguously due to a rise in Y's price, and not perhaps due also to a fall in X's price (or to a rise in X's price less steep than that in Y's price).

In order to concentrate on these problems, Sraffa makes a drastic series of simplifications in his next model, which I shall call $S2$. He retains the essentials of model $S1$, modifying only Assumption 3 in order to permit the production of each commodity to be such that there might be a *surplus* over and above the

[1] Let us observe, in this connection, that in a footnote of his 1945 explication of the von Neumann model, D. G. *Champernowne* acknowledged the help derived from discussions with Sraffa (Review of Economic Studies 13, 1945–1946, p. 10). This note is, incidentally, useful in indicating the influence of the von Neumann model on the various main streams (*Champernowne*, *Kahn*, *Kaldor*, Joan *Robinson*, *Sraffa*) that make up current Cambridge thought on the theory of capital.

[2] The Works and Correspondence of David Ricardo, ed. P. *Sraffa*, vol. I, Cambridge, 1951, pp. xxx–xxxiii. The logical structure of the Ricardian system has recently been brilliantly presented by L. L. *Pasinetti* (Review of Economic Studies 27, 1959–1960, pp. 78–98); the interested reader is recommended to study this article before attempting Sraffa's book, since it provides an excellent account of the Ricardian modes of thought immanent in Sraffa's approach.

inter-industry demands for it as input. Once this is done, and once we abandon the Malthusian wage doctrine (at least in its strict form), it becomes necessary to abandon the idea of labor being produced, like any other commodity, by recipe. This could, of course, be taken care of by a theory accounting for the distribution of the national income between wages and profits[1].

But this would not fit in with Sraffa's main object, which is to see what happens to prices and the *rate* of profit when the allocation between wages and profits is varied. To use a simile, it is as if we varied the proportion ω of income going to wages (with a corresponding proportion $(1-\omega)$ going to profits) by varying the resistance in a simple electrical circuit. We could then imagine observing on suitable meters the varying price of each commodity, together with the varying rate of profit.

It is important to bear in mind always that, as Joan *Robinson* says (op. cit., p. 54), Sraffa's $S2$ is only "half of an equilibrium system". What we are given are the production recipes of $S1$, together with the fixed quantities of outputs. We also have *Assumption (4)*, that the rate of profit (defined below) should be the same in each industry. Sraffa gives no justification for this, but it is obviously the equilibrium condition (in a world of certainty) for a dynamic process in which each capitalist tries to maximize his profits.

What $S2$ does *not* grant is the possibility of making any variation in either output scale or input proportions. Capitalists receive surplus, but do not invest it in the further expansion of outputs; nor, apparently, do they consume it, at least in the sense that there exist no demand equations for either capitalists or workers. Since we are mainly concerned with prices, there seem to be three possible alternative assumptions by which this procedure might be rationalized:

(i) Constant returns to scale exist in each industry, which implies abandoning the fixity of the given outputs. Sraffa says in his preface that this is a harmless provisional assumption for the reader to make, but that in fact "no such assumption is made"; we shall return to this point later.

(ii) Following Mrs. *Robinson's* suggestion (op. cit., p. 54), no variation in the division between wages and profits can alter the commodity composition of output. This is an extremely restrictive assumption to swallow.

(iii) Some central mechanism exists for allocating the *commodity* composition of profits and wages, subject to the restraints of $S2$. This is necessary, since this commodity composition is not otherwise determined by the system, except in the two industry case, as in Sraffa's example on p. 7. But since there is then no possibility of trade among consumer goods, this third assumption seems to rule out any rationale for equal money profit rates (and wage rates) in each industry.

[1] Land, and hence land rent, is not introduced until Part II.

Suppose that a vegetarian capitalist were allotted his profits in the form of meat; if he cannot trade, it is small consolation to be told that the money value of this meat, in proportion to the money value of the means of production he employs, is the same as that of everyone else's allocation (in proportion to the value of the means of production that *they* command).

All this is only to stress that half of an equilibrium system is just that, and not a small complete system. One wonders a little whether the striking results that Sraffa obtains would survive substantially intact in a more complete model, which would need to include a distribution theory, demand equations and some degree of factor substitution, and which would abandon the assumption of a uniform "harvest period" for each commodity[1]. But it is certainly of importance to investigate the bare logic of $S2$ and its variants, even though the discussion of variations of prices with changes in income distribution (all the while keeping to the same point of the n-dimensional output space), reminds one at times of the medieval scholastic debates concerning angels dancing on the point of a pin.

III.

Let us keep to the notation of I, and replace Assumption (3) by (3a), i.e., that $\sum_i a_{ij} \leqq 1$ for each j, with $\sum_i a_{ij} < 1$ for at least one j. We assume that each good is used in the production of every other good, either directly or indirectly. Define the rate of profit r_i in the ith industry by

$$r_i = \frac{p_i - \sum_j a_{ij} p_j}{\sum_j a_{ij} p_j}. \tag{3i}$$

By Assumption (4), r_i is independent of i, and all rates of profit are equal to the uniform rate of profit r, which may be written

$$r = \frac{\sum_i p_i - \sum_i \sum_j a_{ij} p_j}{\sum_i \sum_j a_{ij} p_j} = \frac{\sum_i p_i - V}{V}. \tag{3ii}$$

The first term on the right hand side of (3ii) is simply the value of total product, and the other term V is the total value of the means of production. By Assumption (4), this latter quantity is raised in each industry by the expansion factor $(1+r)$ and the matrix equation for $S2$ becomes

$$(1+r)\,\mathbf{Ap} = \mathbf{p}. \tag{4}$$

[1] In Part II Sraffa makes considerable progress in relaxing this last assumption (whose crucial importance for Ricardo is well brought out by *Pasinetti,* op. cit., p.91), by his very clever treatment of fixed capital in terms of joint products, a device due originally to Torrens, and exploited in particular by *von Neumann.*

Setting $c = \frac{1}{(1+r)}$, (4) becomes

$$\mathbf{Ap} = c\mathbf{p}. \tag{4i}$$

Assuming for the moment that labor is a product like any other, we may ask if there is a solution to (4i) giving a positive price vector and a positive rate of profit. The answer is yes, and here we call on a remarkable theorem, due essentially to Perron[1], and much used recently in mathematical economics. Since **A** is a matrix which is non-negative and indecomposable (a term defined below), there will by this theorem exist a real, positive root c^* of **A**, which is no less in absolute value than any other root of **A**, and which is unique (i. e. all other real positive roots are smaller). Moreover, with c^* is associated an eigenvector $\mathbf{p}^*$, each of whose components is non-zero and of the same sign. Since $\mathbf{p}^*$ is unique only up to a scale factor, we may always normalize it so that $\mathbf{p}^*$ is a positive vector. Hence there exists $\mathbf{p}^* > 0$ such that

$$\mathbf{Ap}^* = c^*\mathbf{p}^*. \tag{4ii}$$

Sraffa chooses to normalize p^* by the condition that the money value of the national income be unity (p. 12), which may be written

$$\mathbf{s}'\mathbf{p}^* = 1, \tag{5}$$

where **s** is the n-dimensional column vector of the physical surpluses s_j in each industry, and $\mathbf{s}'$ denotes the row vector which is the transpose of **s** (i. e. $s_j = 1 - \sum_i a_{ij}$, so that $s_j \geqq 0$ and $\mathbf{s}'$ is a semi-positive vector).

As yet, however, we have only proved that the root c^* is positive; what we need to prove is that the equilibrium rate of profit r^* is positive. Since $c^* = \frac{1}{(1+r^*)}$, the requirement that r^* be positive (and finite) implies that c^* must be less than 1 and greater than zero. Fortunately this further result is also a corollary of the Perron theorem. Let M be the largest of the numbers $\sum_i a_{ij}$ (i. e. the largest column sum), and let m be the smallest such column sum. Then provided that m is actually less than M, we can assert that

$$m < c^* < M. \tag{6}$$

[1] See e.g., *Gantmacher*, op. cit., Chapter III, pp. 61–79. The theorem quoted is actually due to Frobenius, and represents a fairly straightforward extension of the basic result for positive matrices due to Perron. It may also be shown that no other root of **A** has a one-signed eigenvector associated with it, so that the rate of profit determined by c^* is the *only* rate which is consonant with positive prices.

Sraffa does not, at this stage, provide a proof of positivity of $\mathbf{p}^*$ and r^*; for his later discussion of this point, see Section *V* below.

Since none of the column sums of **A** is greater than 1, and at least one is less (by Assumption (3 a)), while all column sums are positive, it follows that c^* must be between 0 and 1; if all column sums are equal, then $m = M = c^*$. Hence r^* is positive in any event. This proof also demonstrates the reasonable proposition that the equilibrium rate of profit must lie between the greatest and the smallest industry rate of surplus (measured as ratios of physically measurable quantities) in the system. Clearly r^* and $\mathbf{p}^*$ depend only on the "recipe" matrix **A**.

In proving those results we have assumed that: (a) each commodity is used as input, either directly or indirectly, in the manufacture of all other commodities (e.g., iron used in steel, steel used in ships, ships used in freight service, freight service used in wheat, wheat used in bread, and so on, where the chain would run through all commodities); and (b) that labor consumes fixed levels of inputs, irrespective of the rate of profit. Let us now relax these assumptions, beginning with (a). Sraffa calls those commodities which do not enter into the production of any other commodity, "non-basic commodities" (he later generalizes this definition considerably). Suppose that we have a two-sector model (think of Ricardo's "corn" and "iron"), and let the first commodity–"iron"–not enter into corn production, so that iron is "non-basic". This means that $a_{21} = 0$, and the equations corresponding to (4) will then be

$$\begin{aligned} (1+r)\, a_{11} p_1 + (1+r)\, a_{12} p_2 &= p_1, \\ (1+r)\, a_{22} p_2 &= p_2, \\ (1-a_{11})\, p_1 + (1-a_{12}-a_{22})\, p_2 &= 1. \end{aligned} \tag{7}$$

Since the unknowns are p_1, p_2 and r, it might be thought that the system (7) is sufficient to ensure the existence of a solution. But remember that the solution must contain *positive* prices and a *positive* uniform rate of profit. Let us take a particular numerical example of (7), as follows:

$$\begin{aligned} (1+r)\, 0.8 p_1 + (1+r)\, 0.3 p_2 &= p_1, & \text{(i)} \\ (1+r)\, 0.2 p_2 &= p_2, & \text{(ii)} \\ 0.2 p_1 + 0.5 p_2 &= 1. & \text{(iii)} \end{aligned} \tag{7 a}$$

Now if $p_2 \neq 0$, we can conclude from (ii) that $(1+r) = \dfrac{1}{a_{22}} = 5$. Substituting this into (i), we obtain $p_2 = -2p_1$, which from (iii) yields $p_1 = -\frac{5}{4}$, $p_2 = \frac{5}{2}$, and $r = 4$. Hence if $p_2 \neq 0$, the solution contains a negative price. If $p_2 = 0$, then $p_1 = 5$ and $r = \frac{1}{4}$. In either case we have a contradiction of Sraffa's combined requirements that the system be in a self-replacing state and that profit rates be uniform. If the value of p_2 is negative, then even though it uses only itself in its own production, the fact that $a_{22} < 1$ means that production of corn results in negative profit. If $p_2 = 0$, then its production cannot add to profit. Since there are positive profit opportunities in "iron", corn is not produced at all. But then iron production must cease also—since it requires corn—and the system is not self-replacing in any part.

It is possible to find necessary and sufficient conditions that a matrix containing "non-basics" will always yield prices which are all positive[1], but these conditions appear to have little economic significance; in the present very simple example, they specialize to $a_{11} < a_{22}$, the economic rationale for which seems obscure. Therefore either we must abandon one of Sraffa's Assumptions (1)–(4), or we must assume that "non-basics" do not exist, if we are not to confine ourselves to a rather odd and restricted class of situations. I shall choose the course of abandoning "non-basics", since Assumptions (1)–(4) are the crux of Sraffa's system. This choice is reinforced by the consideration that the question of whether a good is "non-basic" is partly a matter of the degree of aggregation in the system. Thus suppose that initially we have a three sector system as follows:

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{matrix}$$

in which the first good is non-basic. If we now aggregate the first and second sectors to one, we will obtain a new matrix

$$\begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{matrix}$$

in which there is no non-basic commodity.

This result, that non-basics will often not imply a positive price vector, means that the rather heavy emphasis placed on such commodities by Sraffa (he exemplifies them by luxury goods), seems misplaced. We may welcome the result, since it greatly simplifies the following analysis.

[1] *Gantmacher*, op. cit., Theorem 6, p. 92. Considering now Sraffa's *generalization* of "non-basics" (which are also excluded by the considerations in the text), we may say that if there are non-basics, the matrix **A** is *decomposable*, i.e., by suitable interchanging of rows and columns, it can be put in the form

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_3 \end{bmatrix}$$

where $\mathbf{A}_1$ and $\mathbf{A}_3$ are square matrices, and $\mathbf{0}$ is a matrix of zeros. For a discussion of the economic aspects of such matrices, see e.g. *Dorfman*, *Samuelson* and *Solow:* Linear Programming and Economic Analysis (McGraw-Hill, 1958), Chapter 10. If a matrix is not decomposable, it is *in*decomposable, which is equivalent to containing no "non-basic" commodities.

IV.

The relaxation of the second provisional assumption, that there are fixed inputs into labor, poses such important problems that it occupies the rest of Part I. Sraffa argues that when there is surplus in the system, it is only reasonable to expect that labor will share in it. He wistfully lingers over the possibility of dividing the wage into an "inter-industry" and a "surplus" part, but rejects it in order to conform to common usage. Such a division would seem reasonable if there were two non-competing groups of laborers–say slaves and freemen–but is (as Mrs. *Robinson* remarks p. 54) very artificial otherwise. The model $S2$ was a genuine slave economy, since capitalists made profit at the usual rate on the production of labor, but in the new model, $S3$, there is no slave production. The total quantity of labor available is given (at 1 unit), and persists through time unchanged and unresponsive to prices or wages. However, we are still given quantities (this time $m(=(n-1))$ in number) which tell us the proportion of labor (l_i) used in the production of the unit level of the ith good; the vector of the l_i will be denoted by $\mathbf{L}$.

The new system involves an m-dimensional technology matrix $\mathbf{B}$, and corresponding price and "surplus" vectors, $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ respectively. Let us denote the amount of the national income going to wages by W, and the *proportion* going to wages by ω. It is assumed implicitly that the wage rate is the same in each industry, although similar criticisms can be levelled at that as were done at the assumption of equal profit rates in Section II. With these assumptions, and following the normalization rule (5), the system $S3$ may be written

$$\begin{aligned} (1+r)\mathbf{B}\boldsymbol{\pi} + W\mathbf{L} &= \boldsymbol{\pi}, &\quad \text{(i)} \\ W &= \omega \boldsymbol{\sigma}' \boldsymbol{\pi}, &\quad \text{(ii)} \\ \boldsymbol{\sigma}'\boldsymbol{\pi} &= 1. &\quad \text{(iii)} \end{aligned} \qquad (8)$$

I shall postpone until the next section a consideration of whether there always exists a positive solution for r and $\boldsymbol{\pi}$, given any pre-assigned value of ω (between 0 and 1), and shall simply assume provisionally that such a solution exists.

Since there is a fixed amount of labor, a reduction in ω corresponds to a fall in the wage rate. Industries that use a *relatively* small amount of labor will be worse off than those using a large amount, although complications arise since the means of production used by an industry of low labor-intensity may themselves have high labor-intensity. In any event, relative prices will almost certainly have to change if the system is to continue unperturbed, with a uniform rate of profit[1].

[1] There is a close similarity in method of approach here between Sraffa and Joan *Robinson*'s "Rising Supply Price", Economica 1941, reprinted both in Readings in Price Theory (ed. *Boulding* and *Stigler*), and in Mrs. *Robinson*'s Collected Economic Papers Vol. 1 (Blackwell, 1951). Sraffa does not refer to this paper nor, indeed, to any literature of post-World War I vintage.

Suppose, however, that there were an industry which employed labor and other means of production in such proportion that with a change in ω, the balance between wages and profits were maintained at the original level. This would imply that such a relationship held for *each* of the industries supplying the first industry's means of production; and so on back to the whole set of industries which directly or indirectly supply the first. Obviously the existence of such an industry would be very unlikely, but perhaps it could be *constructed,* as a "composite commodity", from the others.

Observe that the price of such an industry's product would never rise or fall with changes in ω, since there would never at any time be a change in the wage-profit proportion (for the full reasoning, see Sraffa's Chapter III). It could therefore serve as a standard commodity, in terms of which all other prices could be measured. Sraffa then tackles the problem of whether such a commodity could always be constructed. I shall not follow his rather unconvincing proof of this (pp. 26–27), since an easier and much more illuminating route is available.

Sraffa starts his analysis by posing an apparently different question. Can we, by taking appropriate positive "fractions" of existing industries, construct a system such that the proportionate excess of output of each commodity, over the amount used as means of production in the system, is the same for each industry? Remembering that each output is 1 unit, this may be expressed by asking if there is a positive vector of numbers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that

$$\frac{\lambda_1}{\sum_i \lambda_i b_{i1}} = \frac{\lambda_2}{\sum_i \lambda_i b_{i2}} = \ldots = \frac{\lambda_m}{\sum_i \lambda_i b_{im}} = \frac{1}{v} \tag{9}$$

say, where all summation signs, here and in the sequel, are over the $(n-1)$ integers $1, 2, \ldots, m$.

We may write (9) as $\sum_i \lambda_i b_{ij} = v\lambda_j \qquad (j = 1, 2, \ldots, m)$ (9a)

or, in matrix form, where $\boldsymbol{\lambda}'$ is the *row* vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$

$$\boldsymbol{\lambda}'\mathbf{B} = v\boldsymbol{\lambda}'. \tag{9b}$$

Taking the transpose of (9b), we get

$$\mathbf{B}'\boldsymbol{\lambda} = v\boldsymbol{\lambda} \tag{10}$$

which is of exactly the same form as (4i). $\mathbf{B}'$ is a non-negative indecomposable matrix, and its roots are the *same* as those of $\mathbf{B}$. Hence the dominant root v^* of $\mathbf{B}'$, which gives a positive equilibrium vector $\boldsymbol{\lambda}'^*$, is the *same* as the maximal root of $\mathbf{B}$, which is $c^* = \dfrac{1}{(1+r^*)}$.[1]

[1] Sraffa asserts that if λ' is to be positive (he uses q's rather than λ's), then no non-basic can enter the system. This is not quite correct, as we have seen earlier (footnote on

From a mathematical point of view, $\boldsymbol{\lambda}'^*$ is the *row* eigenvector corresponding to the maximal root c^* of $\mathbf{B}$, just as $\boldsymbol{\pi}^*$ (the price vector when all surplus goes to capital) is the *column* eigenvector corresponding to c^*. It follows that $\boldsymbol{\lambda}'^*$ is unique up to a normalizing factor, which Sraffa obtains by the relation

$$\boldsymbol{\lambda}'^*\mathbf{L} = 1. \tag{11}$$

This means that we take the *standard system* to be that which uses all the available labor supply of the *actual* system. Such a system produces the *standard national income*, and the "standard" ratio R of this income to the "standard" means of production is given, for *any* price vector $\boldsymbol{\pi}$, by

$$R = \frac{\sum_j \pi_j (\lambda_j - \sum_i \lambda_i b_{ij})}{\sum_j \pi_j \sum_i \lambda_i b_{ij}} \tag{12}$$

which, from (9a), assumes the form

$$R = \frac{(1-v^*) \sum_j \pi_j \lambda_j}{v^* \sum_j \pi_j \lambda_j} \tag{12a}$$

Remembering that $v^* = c^* = \dfrac{1}{(1+r^*)}$, and since each π_j and λ_j in positive, (12a) reduces to

$$R = \frac{1-c^*}{c^*} = r^*. \tag{13}$$

Hence the standard ratio is (i) equal to the rate of profits of the actual system when none of the surplus goes to labor[1] (a rate which Sraffa calls the Maximum Rate), and (ii) is independent of prices, depending only on $\mathbf{B}$, the technology matrix. It follows that this standard commodity has the desired properties laid down earlier.

We may pause here to enquire whether (following Sraffa) we have not implicitly smuggled an assumption of constant returns to scale into the analysis here. For we have multiplied, in each industry, each of the inputs by the fraction λ_i, and have assumed that output will now be λ_i. One could argue in defense, at least up to equation (12), that this trick has merely been a computing device to enable us to find the appropriate vector $\boldsymbol{\lambda}'^*$, given $\mathbf{B}$, and that no changes in output actually occur. This is a little harder to maintain with equation (12), for then we actually consider what the "standard" level of production would be.

page 10), though the conditions *are* very stringent. However, using a result of *Gantmacher* (p. 96), we can assert that if *both* π and λ are to be positive, $\mathbf{B}$ must be indecomposable, so that there are then no non-basics.

[1] In $S3$, unlike $S2$, labor does not get any reward at all when $\Theta = 0$, since labor is not included in $\mathbf{B}$. This enhances the desirability of considering two non-competing types of labor, a device discussed above.

But do we? We are still dealing only with a *Hilfskonstruktion,* the Standard System, and are not committed to the assertion that if we *actually* changed input levels by a fraction λ_i, we would *observe* output to be changed by the same fraction λ_i. The important point to take hold of is that at no time in Part I do any output levels actually change, so that the question of whether constant returns to scale do or do not prevail can have no meaning.

Thus reassured, let us consider further properties of the standard system. Suppose that a fraction Θ of the standard national income goes to wages. This is equivalent to multiplying the numerator of (12) by $(1-\Theta)$, which leads to the conclusion that the rate of profit r in the standard system will be

$$r = R(1-\Theta). \tag{14}$$

One achievement of Part I is to show that such a linear relation between r and Θ is *not* confined to the standard system, but holds in the actual system, provided that wages are paid in "standard commodity" i.e. in a fixed "basket of goods", with weights $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. Sraffa's proof of this in the book (p. 22) is mostly assertion, but it is fairly easy to demonstrate. The rate of profit r_a in the actual system is the amount of the national income remaining after paying away to wages the monetary equivalent of a fraction Θ of the standard national product, divided by the value of the actual means of production. Thus

$$r_a = \frac{\sum_j \pi_j (1-\sum_i b_{ij}) - \Theta(\sum_j \pi_j (\lambda_j - \sum_i \lambda_i b_{ij}))}{\sum_j \pi_j (\lambda_j - \sum_i \lambda_i b_{ij})}. \tag{15}$$

Since by Assumption (4), r_a is the rate of profit in each industry, we have also (analogously to (3i)),

$$r_a = \frac{\pi_i - \sum_j b_{ij}\pi_j - l_i\Theta(\sum_j \pi_j (\lambda_j - \sum_i \lambda_i b_{ij}))}{\sum_j b_{ij}\pi_j}, \tag{16}$$

$$i = 1, 2, \ldots, m.$$

The last term in the numerator of (16) is the proportion of the total wage bill accruing to the ith industry; the same term does not appear in the denominator because Sraffa assumes (p. 10) that wages are *not* advanced from capital, but paid *ex post* as a share of the national product.

From (16), and the fact that $\lambda_i > 0$ for each i,

$$r_a = \frac{\lambda_i \pi_i + \lambda_i (\sum_j b_{ij}\pi_j) - \lambda_i l_i \Theta(\sum_j \pi_j (\lambda_j - \sum_i \lambda_i b_{ij}))}{\lambda_i \sum_j b_{ij}\pi_j}, \tag{17}$$

$$i = 1, 2, \ldots, m.$$

Summing over all i,

$$r_a = \frac{\sum_i \lambda_i \pi_i - \sum_j \pi_j (\sum_i \lambda_i b_{ij}) - \Theta\sum_i \lambda_i l_i (\sum_j \pi_j (\lambda_j - \sum_i b_{ij}))}{\sum_j \pi_j (\sum_i \lambda_i b_{ij})}. \tag{18}$$

From (9a) and (11), and because $v^* = c^*$,

$$r_a = \frac{\sum_i \lambda_i \pi_i - c^* \sum_j \pi_j \lambda_j - \Theta(\sum_i \pi_i \lambda_i - c^* \sum_j \lambda_j \pi_j)}{c^* \sum_j \pi_j \lambda_j}. \tag{19}$$

Since in this equilibrium situation each π_i and λ_i is positive, (19) reduces to

$$r_a = \frac{(1-c^*)(1-\Theta)}{c^*}$$

or

$$r_a = R(1-\Theta). \tag{20}$$

Notice that Θ is the fraction of the *standard* national income that goes to wages, and not the fraction ω of the actual national income. Sraffa adopts a new normalization rule for $\boldsymbol{\pi}$, using

$$\sum_j \pi_j (\lambda_j - \sum_i \lambda_i b_{ij}) = 1 \tag{21}$$

which from (9a) can be written

$$(1-c^*) \sum_j \pi_j \lambda_j = 1 \tag{22}$$

or in matrix form, from (13)

$$\boldsymbol{\lambda}' \boldsymbol{\pi} = \frac{R+1}{R}. \tag{23}$$

V.

Our remaining task, left over from the early part of Section IV, is to prove that the system $S3$, given by

$$\begin{aligned} (1+r)\mathbf{B}\boldsymbol{\pi} + W\mathbf{L} &= \boldsymbol{\pi}, && \text{(i)} \\ W &= \omega \boldsymbol{\sigma}' \boldsymbol{\pi}, && \text{(ii)} \\ \boldsymbol{\sigma}' \boldsymbol{\pi} &= 1, && \text{(iii)} \end{aligned} \tag{8}$$

always has a positive solution for the price vector $\boldsymbol{\pi}$, whatever the value of the wage-share parameter ω. As Sraffa remarks however (p. 33), once we cease to regard wages as physiologically given, it becomes less natural to suppose that it is ω which should be independently varied. To quote him: "The rate of profits, as a ratio, has a significance which is independent of any prices, and can well be 'given' before the prices are fixed. It is accordingly susceptible of being determined from outside the system of production."

Therefore we shall investigate the positivity of prices, profits and wages as the *rate* of profits r is varied[1]. A valid proof is not easy to find, but can be con-

[1] Sraffa's proof of this (pp. 27–28) is deficient, mainly because it assumes that wages and profits remain positive throughout; since these are partly the resultant of prices, the argument appears to be circular.

This is one example of the inadequacy of several of Sraffa's proofs. Since most of his theorems are essentially correct, it is an open question whether Sraffa in fact (perhaps structed with the help of material contained in a remarkable book by the mathematician Jacob *Schwartz*[1], which appeared at a late stage in the preparation of this article. We begin by observing that since $\boldsymbol{\sigma}'$ is semi-positive, (8iii) implies that a price vector which is at least semi-positive must exist. Since this seems to prejudge the issue, let us reduce $S3$ to (8i) and (8ii) only; we can add (8iii) at such time as we have proved that $\boldsymbol{\pi}$ is positive.

It will prove convenient to rewrite (8i) in a new form, by writing $P_i = \frac{\pi_i}{W}$ for each i, and so obtaining

$$\mathbf{L} = [\mathbf{I} - (1+r)\mathbf{B}]\mathbf{P} \tag{24}$$

where $\mathbf{I}$ is the m-dimensional diagonal unit matrix, and $\mathbf{P}$ is the column vector of the P_i. Now in view of (8ii), we can only write (24) if ω is non-zero and the scalar $\boldsymbol{\sigma}'\boldsymbol{\pi}$ is non-zero, for otherwise W will be zero. If ω is zero, then (8i) reduces to an m-dimensional version of (4), for which we know there exist positive solutions for r and $\boldsymbol{\pi}$.

If $\boldsymbol{\sigma}'\boldsymbol{\pi}$ is zero, however, the problem is more delicate. It follows from (3ii) that r must then be zero unless V is zero, in which case r assumes the indeterminate form $\frac{0}{0}$. Assuming temporarily that this is not the case, the conditions $r = 0$ and $W = 0$ imply that (8i) reduces to

$$\mathbf{0} = (\mathbf{I} - \mathbf{B})\boldsymbol{\pi} \tag{25}$$

where $\mathbf{0}$ is a column vector of zeros. Since $(\mathbf{I}-\mathbf{B})^{-1}$ exists (actually consisting entirely of positive elements)[2], it follows that $\boldsymbol{\pi}$, which is equal to $(\mathbf{I}-\mathbf{B})^{-1}\mathbf{0}$, is also a vector of zeros. If, instead, V in (3ii) is also zero, we fall back on the fact, due to Assumption 4, that r must also satisfy each of the $(m-1)$ equations (3i). Since $\boldsymbol{\sigma}'\boldsymbol{\pi}$ and W are both zero, those equations can clearly only be satisfied if each price is zero.

Hence $\boldsymbol{\sigma}'\boldsymbol{\pi}$ only vanishes for the trivial null solution $\boldsymbol{\pi} = \mathbf{0}$; it follows that we are always entitled to write equation (24). We may now pose the problem

with the help of *Besicovitch, Ramsey* et al.) has more adequate proofs up his sleeve, proofs which he did not include for fear of making the book "too mathematical". Given the present temper of pure economic theory, such a fear seems misplaced; their inclusion (should they exist) would have made the book *less* difficult.

[1] Jacob L. *Schwartz:* Lectures on the Mathematical Method in Analytical Economics (Gordon and Breach, New York 1961). It is my impression that Part C of Schwartz's book does, in a sense, carry forward the programme announced in Sraffa's preface, of preparing a systematic critique of neo-classical general equilibrium theory, a critique based on Sraffa-type models. The similarity in methods of approach is striking, especially in their similar conclusions regarding the relatively minor role in price formation played by individual demand functions. Since Schwartz makes no reference to Sraffa, this may be taken as supporting evidence for the considerable originality displayed in the former's work.

[2] This is another consequence of the Perron theorem. See e.g. G. *Debreu* and I.N. *Herstein:* Non-negative Square Matrices, Econometrica 21, 1953, Theorem III, p. 602.

of this section by asking if there exist positive (i.e. every element > 0) inverses of the one-parameter family of matrices $(\mathbf{I} - (1+r)\mathbf{B})$, for a sufficiently large interval of values of the parameter r.

If so, then the equation

$$\mathbf{P} = (\mathbf{I} - (1+r)\mathbf{B})^{-1}\mathbf{L} \tag{26}$$

assures us that $\mathbf{P}$ will be a positive vector. Since (8ii) may be written

$$1 = \omega \boldsymbol{\sigma}' \mathbf{P} \tag{27}$$

it follows, by multiplying both sides of (27) by $W (\neq 0)$, that the positivity of $\mathbf{P}$ implies that W and $\boldsymbol{\pi}$ are both positive, and hence total profits also[1].

An affirmative answer to our question can be derived straightforwardly from a series of results in Schwartz's Lectures 2 and 3 (op. cit.), provided that we set *his* matrix $\boldsymbol{\pi}$ equal to his $\boldsymbol{\Phi}$, and identify both with our $\mathbf{B}$. Let us denote the maximum, or dominant, latent root of $(1+r)\mathbf{B}$ by the symbol dom $(1+r)\mathbf{B}$. Since $\mathbf{B}$ is a positive matrix, $(1+r)\mathbf{B}$ will be positive for $r > -1$, and hence dom $(1+r)\mathbf{B} > 0$ for $r > -1$.

From Schwartz's lemma 3.3, it follows that an inverse will always exist if r is such that dom $(1+r)\mathbf{B} < 1$. Suppose that r_{max} is the least upper bound of those r's for which this is true i.e. dom $(1+r)\mathbf{B} < 1$ for $r < r_{max}$. By Theorems 2.2 and 2.3 of his Lecture 2, it follows that dom $(1+r)\mathbf{B}$ is a strictly increasing and *continuous* function of r; hence we can equivalently define r_{max} as the unique solution of the equation

$$\text{dom } (1+r_{max})\mathbf{B} = 1. \tag{28}$$

An alternative way of expressing this is to say (remembering equations (4) to (4ii)) that r_{max} is the unique number for which there exists a positive solution vector $\mathbf{P}_{max}$ of the system

$$(1+r_{max})\mathbf{B}\mathbf{P} = 1\mathbf{P} \tag{29}$$

which may be written

$$\mathbf{B}\mathbf{P} = \frac{1}{(1+r_{max})}\mathbf{P}. \tag{30}$$

But (30) is just another way of writing the solution to (8i) when ω, and hence W, $= 0$. Therefore r_{max} is simply our old friend the Maximum Rate of Profit R, which obtains when the wage-share ω is zero. Moreover we know that since dom $\mathbf{B} < 1$, $r_{max} > 0$, so that the open interval $(-1, r_{max})$ is not empty. It follows that for $-1 < r < r_{max}$ we have dom $(1+r)\mathbf{B} < 1$, which in turn implies that $(\mathbf{I} - (1+r)\mathbf{B})^{-1}$ exists for all such r. Since $(1+r)\mathbf{B}$ is a positive

[1] Strictly speaking our argument only shows that W and each price must have the same sign, which could be negative. But since we have already shown that for $W = 0$, $\boldsymbol{\pi}$ is positive, it follows by simple continuity arguments that the common sign must in fact be positive.

matrix for this range of values of r, it follows from the result quoted for equation (25) that the inverses not only exist but also consist entirely of positive elements.

Hence $\mathbf{P}$ is a positive vector, which is our required main result. Utilizing Schwartz's lemma 3.6, we can assert an even stronger result. This lemma enables us to say that if $(1+r_1)\mathbf{B} < (1+r_2)\mathbf{B}$, dom $(1+r_1)\mathbf{B} < 1$ and dom $(1+r_2)\mathbf{B} < 1$, then

$$(\mathbf{I}-(1+r_1)\mathbf{B})^{-1} < (\mathbf{I}-(1+r_2)\mathbf{B})^{-1}. \tag{31}$$

Now if $-1 < r_1 < r_2 < r_{\max}$, (31) applies, so that we can say that the solution vector $\mathbf{P}$ is a strictly increasing function of r (Schwartz's Theorem 3.7). *All* prices rise in terms of wages as the rate of profits is increased towards $r_{\max}$.

We have spoken of the interval for r as $(-1, r_{\max})$, but there is something repugnant to economic commonsense in supposing that an equilibrium value of r can be less than zero. Accordingly we restrict the interval to be $(0, r_{\max})$, which is not empty, by the results above. Also, utilizing $S3$ and an equation analogous to (3ii), we can write

$$r = \frac{\boldsymbol{\sigma}'\boldsymbol{\pi}(1-\omega)}{V}. \tag{32}$$

When $r = 0$, then (32) means that $\omega = 1$, so that a zero profit rate implies that all income goes to labor, a reasonable result.

This completes our analysis of Sraffa's Part I[1]; all of his main results have been shown to be valid, and it is a reasonable presumption that the more extended analysis of Part II rests on secure foundations. Whether the work of the book as a whole is considered important depends partly upon the reader's view of pure economic theory. My own view would be that although particular points on which Sraffa lays stress, such as non-basics and the standard commodity, are of greater mathematical interest than economic, the book has made a serious contribution to a re-examination of our theory of general equilibrium. Such work as that of Sraffa and of Schwartz helps us realize that neo-classical Walrasian theory is not *the* general equilibrium theory, but only a *model* of general equilibrium. Other models may be much more helpful for the elucidation of important unresolved problems, especially in dynamics.

[1] The last chapter of Part I is a straightforward "reduction" of prices to dated quantities of labor, a process familiar from Leontief models in which labor is the only primary input. A somewhat similar analysis is to be found in Section 3 of Schwartz's Lecture 4.

Further results from the latter's Lecture 3 would enable us to carry through a similar analysis to that in the text for the case where there are several grades of labor, instead of just one.